Number Theory 18

14 March 2024 18:04

Q) Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number. $(a+1) \mid (b-1)$ and $b \mid (a^2+a+2)$.

Ans:— $b-1 = (a+1)\underbrace{k_1}_{k_1 \geq 0} \Rightarrow (a+1)k_1+1 \mid a(a+1)+2$

$\Rightarrow a(a+1)+2 \equiv 0 \pmod{(a+1)k_1+1}$

$\Rightarrow (a+1)(a-2k_1) \equiv 0 \pmod{(a+1)k_1+1}$

We know $a+1$ and $(a+1)k_1+1$ are coprime

$\Rightarrow (a-2k_1) \equiv 0 \pmod{(a+1)k_1+1}$

either

$\Rightarrow a-2k_1 \geq ak_1+k_1+1$ but $a-2k_1 \leq a$ as $k_1 \geq 0$ and $ak_1+k_1+1 \geq a$

$\Rightarrow a-2k_1 \leq ak_1+k_1+1$

$\Rightarrow a-2k_1 = ak_1+k_1+1$

$\Rightarrow a(1-k_1) = 3k_1+1$

$\Rightarrow a = \frac{3k_1+1}{1-k_1}$ but $k_1 \geq 0$

if $k_1 = 0$ then $a = \frac{3}{1} = 3$

$k_1 = 1$ then $a =$ undefined

$k_1 > 1 \Rightarrow a < 0 \Rightarrow\Leftarrow$

no need to solve, no solution

or

$2k_1 - a \geq ak_1+k_1+1$

$2k_1 - a \leq ak_1+k_1+1$

$\Rightarrow 2k_1 - a = ak_1+k_1+1$

$\Rightarrow k_1 - 1 = a(k_1-1)$

$\Rightarrow a = 1$, if $k_1 \neq 1$

if $k_1 = 1 \Rightarrow 2-a \geq a+2$ (not possible)

no solution

Another case is $2k_1 - a = 0 \Rightarrow a = 2k_1$

$\Rightarrow (b-1) = (2k_1+1)k_1$

$\Rightarrow b = 2k_1^2+k_1+1$

We know that b is a 3 digit number

$\Rightarrow 100 \leq 2k_1^2+k_1+1 \leq 999$

$\Rightarrow \frac{100}{2} \leq k_1^2 + \frac{k_1}{2} + \frac{1}{2} \leq \frac{999}{2}$

$\Rightarrow \frac{100}{2} \leq k_1^2 + 2k\frac{1}{4} + \left(\frac{1}{16}\right)^2 + \frac{1}{2} - \frac{1}{16} \leq \frac{999}{2}$

$\Rightarrow \frac{99}{2} + \frac{1}{16^2} \leq \left(k_1 + \frac{1}{4}\right)^2 \leq \frac{998}{2} + \frac{1}{16^2}$

$\Rightarrow \sqrt{\frac{99}{2} + \frac{1}{16^2}} \leq k_1 + \frac{1}{4} \leq \sqrt{\frac{998}{2} + \frac{1}{16^2}}$

$\Rightarrow 7 \leq k_1 \leq 22$

$\Rightarrow$ No. of solutions $= 22-7+1 = 16$

Q) 2016 coins are placed on a table with 50 heads up and remaining tails up. We need to make equal number of heads in G_1 and G_2.

Ans:- 50 heads

G_1 has $2016-50$ coins $= 1966$ coins

G_2 has 50 coins

G_1	G_2
x heads	$50-x$ heads
$1966-x$ tails	x tails

$\rightarrow$ Now flip all coins of G_2 we get x heads and $50-x$ tails

Q) $y \in \mathbb{N}$ is obtained from x by rearranging its digits. Suppose $x+y = 10^{200}$, prove that x is divisible by 10.

Ans:- $x+y = \text{----}000 \rightarrow$ last digit is 0

$10^1 = 10$

$10-1 = \underbrace{9}_{1 \text{ times}}$

If $10 \nmid x$ then $x = \text{--}a_1a_0$ where $a_0 \neq 0$

$x-1+y = \underbrace{99 \text{ --- } 9}_{200 \text{ times}} \rightarrow$ this means at each index the sum is 9 with no carry

$x-1 = d_{199} \text{------} d_0$

$y = b_{199} \text{-----} b_0$

$\Rightarrow \sum_{i=0}^{199}(d_i+b_i) = 9 \times 200$

Last digit of $x-1$ was (a_0-1) as $a_0>0$ $\Rightarrow x = a_{199} \text{-----} a_0$

$\Rightarrow \sum_{i=0}^{199}(a_i+b_i) = \underbrace{200 \times 9+1}$

Not possible as x and y has the same digits $\leftarrow$ This is odd

and so as $\sum_{i=0}^{199}(a_i+b_i) = 2\sum_{i=0}^{199}(a_i) = 200 \times 9+1 \Rightarrow \Leftarrow$

So $10 \mid x$.

Q) Prove that for prime p,

$x^p - x \equiv x(x-1)(x-2) \text{-----} (x-(p-1)) \pmod{p}$ for any $x \in \mathbb{Z}$

Ans:- $x, (x-1), (x-2), \dots, (x-(p-1)) \Rightarrow$ one of them must be divisible by p

$\Rightarrow$ RHS $\equiv 0 \pmod p$

LHS $= x^p - x \equiv 0 \pmod p$ ... [by Fermat's Little Theorem]

$f(x) = x^p - x$

$g(x) = x(x-1)\dots(x-(p-1))$

$\Rightarrow f(x) \equiv g(x) \;\; \forall x \in \mathbb{Z} \pmod p$

$\Rightarrow f(x) - g(x) = 0 \;\; \forall x \in \mathbb{Z} \pmod p$

$\Rightarrow f(x) \equiv g(x) \pmod p$

HomeWork Let $n, p > 1$ be positive integers and p be a prime. If $n \mid p-1$ and $p \mid n^3 - 1$, prove that $4p-3$ is a perfect square.

HomeWork Calculate the last three digit of $2008^{2007^{2006^{\cdots^{2^1}}}}$.